J. Fluid Mech. (1970), *vol.* 43, *part* 3, *pp.* 487–495
Printed in Great Britain

Shock propagation into inhomogeneous media

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(Received 19 December 1969)

An analytic relation is derived for the shock front velocity as a function of the initial parameters (pressure, density, and particle velocity) in a continuous, inhomogeneous medium. This relation was verified experimentally by using it to predict the propagation of a shock wave through a known rarefaction wave.

1. Introduction

The propagation of shock waves into homogeneous gases is well understood, but the more general question of how shocks propagate into inhomogeneous media has not yet been completely answered. In an inhomogeneous medium, the initial pressure, density, and particle velocity each depend on position. The general problem has been solved partially in several previous studies. Chisnell (1955) considered shock propagation into a region of continuously varying density but with constant initial pressure. Ôno, Sakashita & Yamazaki (1958) extended Chisnell's results to include pressure variation as well as density variation, but their results could not be written analytically. Whitham (1958) presented a more general study, and was able to obtain analytic results for shock propagation into a region of continuously varying density and pressure. Whitham's results are not completely general, since he made several assumptions about the non-uniform region. He used the characteristic equation, and assumed that the pressure and density distributions are maintained in equilibrium by a force field or momentum source term. However, in general, the non-uniform region may first be unsteady, and secondly have energy and mass source terms appearing in the equations of motion. Whitham also makes the strong shock approximation so as to ignore the force term in the characteristic equation. In order to avoid these assumptions, a general study should consider the three fluid dynamic quantities as completely independent parameters. In this study, we allow for a variation of all three initial parameters (pressure p_1, density ρ_1, and particle velocity u_1) ahead of the shock, and we derive the velocity of the shock front as an analytic function of p_1, ρ_1, and u_1.

2. General flow field

We consider the propagation of a shock wave through a one-dimensional, continuous, inhomogeneous medium, where the initial density, pressure, and particle velocity each vary smoothly as a function of position. The propagation of the initial shock through this region is well defined and a unique physical process occurs, i.e. the shock velocity remains single-valued throughout the inhomogeneous region.

The shock wave velocity V is, in general, a function of the initial density, initial pressure, initial particle velocity, and the driving mechanism. The shock velocity is completely specified if these variables are given. In most cases, the exact nature of the driving mechanism is not clearly known, and thus the shock velocity is usually treated as an independent parameter. Still, the shock velocity can be written in the functional form,

$$V = V(\rho_1, p_1, u_1, \xi), \tag{1}$$

where ξ corresponds, in some sense, to the driving mechanism. If we assume that the driving mechanism is constant, the rate of change of the shock velocity throughout the inhomogeneous region is given by differentiating (1)

$$\frac{dV}{dX} = \frac{\partial V}{\partial \rho_1}\frac{d\rho_1}{dX} + \frac{\partial V}{\partial p_1}\frac{dp_1}{dX} + \frac{\partial V}{\partial u_1}\frac{du_1}{dX}, \tag{2}$$

where ρ_1, p_1, and u_1 are treated as completely independent parameters. Integrating (2) yields

$$V(X_1) = V(X_0) + \int_{X=X_0}^{X_1}\left(\frac{\partial V}{\partial \rho_1}d\rho_1 + \frac{\partial V}{\partial p_1}dp_1 + \frac{\partial V}{\partial u_1}du_1\right). \tag{3}$$

Thus, the problem of finding the shock velocity as a function of position X in the inhomogeneous region has been reduced to the problem of finding three functions:

(i) $\partial V/\partial \rho_1$ at constant p_1 and u_1.
(ii) $\partial V/\partial p_1$ at constant ρ_1 and u_1.
(iii) $\partial V/\partial u_1$ at constant p_1 and ρ_1.

One of the partial derivatives $\partial V/\partial \rho_1$ has already been worked out by Chisnell, so that we only have to calculate $\partial V/\partial p_1$ and $\partial V/\partial u_1$.

3. Partial derivatives

The mathematical problem of finding

$$\left(\frac{\partial V}{\partial \rho_1}\right)_{p_1, u_1}, \quad \left(\frac{\partial V}{\partial p_1}\right)_{\rho_1, u_1}, \quad \text{or} \quad \left(\frac{\partial V}{\partial u_1}\right)_{\rho_1, p_1}$$

is equivalent to the physical problem of finding the motion of a normal shock wave through a non-uniform, one-dimensional medium of continuously changing density, pressure, or particle velocity. We assume that the initial parameter increases monotonically with distance in a certain region and is uniform outside this region. A plane shock moves in the X-direction through the region $X < X_0$ with constant strength and uniform flow behind it. When the shock passes through the region of the changing parameter, its strength changes and a wave is reflected backwards from it. In addition, the motion of the reflected wave through the non-uniform region generates another 'doubly reflected' wave moving in the same direction as the incident shock. The mathematical complications encountered by considering the doubly reflected wave are enormous, so we make the approximation that this doubly reflected wave can be ignored. Chisnell (1955) and Ohyama (1961) showed that this is a good approximation if either of the following conditions is met: (i) the incident shock can be described by the strong shock equations, or (ii) the gradients are not too large (reflected shock strength p_2/p_1 is less than 1·25).

Following the method of Chisnell, the non-uniform region is regarded as a succession of small density, pressure, or particle velocity discontinuities separated by uniform regions.

(i) First, we outline Chisnell's derivation of $\partial V/\partial \rho_1$. The corresponding physical situation is that of an infinitesimal density discontinuity (figure 1). The contact discontinuity under consideration $\langle 1, 5\rangle$ has no jump in pressure or fluid velocity, although there is an infinitesimal step in the density, such that

$$\rho_3 = \rho_1 + d\rho, \tag{4}$$

while

$$p_3 = p_4, \quad p_1 = p_5, \tag{5}$$

$$u_3 = u_4, \quad u_1 = u_5. \tag{6}$$

The ratio of the pressure in region n to that in region m is given by the strength Z_{nm} of the disturbance. Therefore, the pressure ratio of the incident shock is $Z_{12} = p_2/p_1$. This parameter is introduced into the conservation equations,

$$\begin{gathered} \rho_1 v_1 = \rho_2 v_2, \\ p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2, \\ \frac{g_1}{g_1 - 1}\frac{p_1}{\rho_1} + \tfrac{1}{2}v_1^2 = \frac{g_2}{g_2 - 1}\frac{p_2}{\rho_2} + \tfrac{1}{2}v_2^2; \end{gathered} \tag{7}$$

and these equations are solved in terms of Z:

$$\rho_2 = \rho_1 \frac{\lambda^2 + Z_{12}}{1 + \lambda^2 Z_{12}}, \tag{8}$$

$$u_2 = u_1 + \phi(Z_{12}, p_1, \rho_1), \tag{9}$$

$$V = u_1 + \left[\frac{p_1}{\rho_1}\frac{\lambda^2 + Z_{12}}{1 - \lambda^2}\right]^{\frac{1}{2}}, \tag{10}$$

where

$$\lambda^2 = \frac{g_2 - 1}{g_2 + 1},$$

$$\phi = (Z_{12} - 1)\left[\frac{p_1}{\rho_1}\frac{1 - \lambda^2}{\lambda^2 + Z_{12}}\right]^{\frac{1}{2}},$$

and g is the effective adiabatic constant (Ahlborn & Salvat 1967).

Similar equations hold for the elements of the reflected wave $\langle 2, 3\rangle$, and differ with those for a shock only in third and higher powers of $Z - 1$. Therefore, we use the shock equations (8)–(10) for the small disturbances $\langle 2, 3\rangle$.

Returning to the conditions on the partical velocity and pressure ((5) and (6)), Chisnell obtained

$$\phi(Z_{12}, p_1, \rho_1) - \phi(Z_{23}, p_2, \rho_2) = \phi(Z_{54}, p_5, \rho_5), \tag{11}$$

$$Z_{12} Z_{23} = Z_{54}. \tag{12}$$

Note that the increase in strength of the penetrated shock must be infinitesimal, so that

$$Z_{54} = Z_{12}+dZ. \tag{13}$$

By inserting (12), (13), (8) and (4) into (11), and taking to the first order in smallness in terms of $d\rho$ and dZ, Chisnell obtained

$$\frac{dZ}{d\rho_1} = \left\{\rho_1\left[\frac{2}{Z-1}-\frac{1}{\lambda^2+Z}+\frac{2}{Z-1}\left[\frac{1+\lambda^2 Z}{Z(1+\lambda^2)}\right]^{\frac{1}{2}}\right]\right\}^{-1}, \tag{14}$$

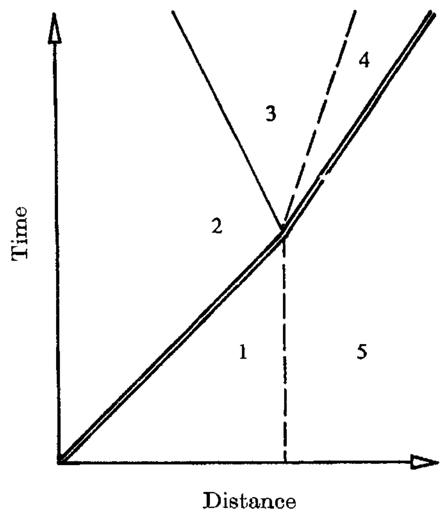

FIGURE 1. When a shock wave $\langle 1, 2\rangle$ crosses an infinitesimal discontinuity $\langle 1, 5\rangle$, it generates at the jump position a refracted shock wave $\langle 4, 5\rangle$, a reflected wave $\langle 2, 3\rangle$, and a contact discontinuity $\langle 3, 4\rangle$.

and, differentiating (10) with respect to ρ_1 at constant p_1 and u_1, he obtained

$$\frac{\partial V}{\partial \rho_1} = \tfrac{1}{2}(V-u_1)\left(\frac{1}{\lambda^2+Z}\frac{dZ}{d\rho_1}-\frac{1}{\rho_1}\right) = f(\rho_1, p_1, u_1, V), \tag{15}$$

where

$$Z = \frac{\rho_1}{p_1}(1-\lambda^2)(V-u_1)^2-\lambda^2. \tag{16}$$

(ii) The physical situation to consider in deriving $\partial V/\partial p_1$ is an infinitesimal pressure discontinuity, across which the particle velocity and density are continuous. Although a rather unnatural type of discontinuity, imagine that some force field maintains it until the shock arrives, after which time the force field is removed. Such a discontinuity could be realized by a soap film which separates sections in the flow. The surface tension of the soap film supports a pressure discontinuity which, for instance, could be produced by heating one side. After the shock has passed, the soap film has evaporated and no surface tension remains.

Again, as in figure 1, an incident shock wave generates at the jump position $\langle 1, 5\rangle$, a reflected wave $\langle 2, 3\rangle$, a penetrated shock $\langle 4, 5\rangle$, and a contact surface $\langle 3, 4\rangle$. The physical quantities before and behind the shock are again connected by the Rankine–Hugoniot equations (8)–(10). Across $\langle 1, 5\rangle$ we have

$$u_1 = u_5, \quad \rho_1 = \rho_5, \quad p_5 = p_1+dp; \tag{17}$$

and, as previously, similar equations describe the particle velocity, and pressure:

$$Z_{12}Z_{23} = Z_{15}Z_{45}, \tag{18}$$

$$\phi(Z_{12}, p_1\rho_1) - \phi(Z_{23}, p_2, \rho_2) = \phi(Z_{54}, p_5, \rho_5). \tag{19}$$

Taking (18) to first order in dZ and dp_1, we obtain

$$Z_{23} = 1 + \frac{dp_1}{p_1} + \frac{dZ}{Z}. \tag{20}$$

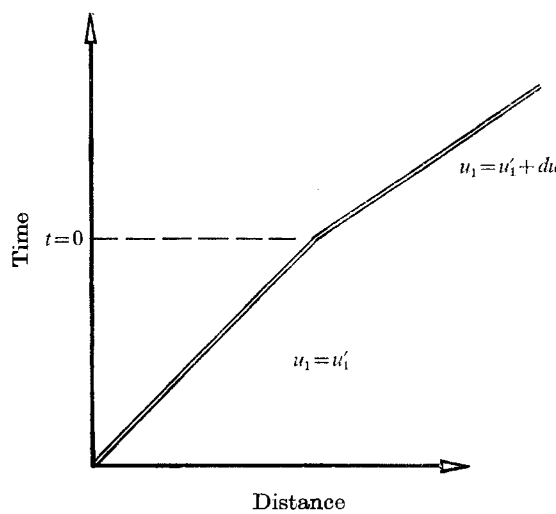

FIGURE 2. At time $T < 0$, the observer sees a shock propagating at a speed V' with strength parameters Z' and ρ_1'/ρ_1 into a gas with particle velocity u_1'. At time $T > 0$, the observer sees a shock propagating at a speed $V + dV$ with the same strength parameters into a gas with particle velocity.

By inserting (20) and (17) into (19), we obtain

$$\frac{dZ}{dp_1} = \frac{-\left[Z\dfrac{1-\lambda^2 Z}{1+\lambda^2}\right]^{\frac{1}{2}} - \dfrac{Z-1}{2}}{p_1\left(1 + \left[\dfrac{1+\lambda^2 Z}{Z(1+\lambda^2)}\right]^{\frac{1}{2}} - \dfrac{Z-1}{2(\lambda^2+Z)}\right)}; \tag{21}$$

and, again differentiating (10) with respect to p_1 at constant ρ_1 and u_1, and introducing (16), we obtain

$$\frac{\partial V}{\partial p_1} = \frac{V-u_1}{2}\left\{\frac{1}{p_1} + \frac{1}{\lambda^2+Z}\frac{dZ}{dp_1}\right\}$$
$$= h(\rho_1, p_1, u_1, V). \tag{22}$$

Equation (22) has been derived in a manner very similar to Chisnell's derivation of $\partial V/\partial \rho_1$.

(iii) Finally, the physical situation, corresponding to the partial derivative $\partial V/\partial u_1$, is a discontinuity across which only the initial particle velocity changes. We note that u_1 is the particle velocity relative to the frame of the observer.

Therefore, an analogous experimental situation would be for an observer to watch the propagation of a shock wave first from some inertial frame at which $u_1 = u_1'$. The observer sees the shock propagate with some velocity V' while the shock strength is Z' and the compression is ρ_2'/ρ_1. Then, at time $T = 0$, the observer changes to a different frame, one in which the particle velocity is $u_1 = u_1' + du$, where du is non-relativistic.

To our observer, the flow of the shock wave appears as in figure 2. Since nothing has happened to the flow itself while the observer has changed frames, the shock will have the same strength parameters Z' and ρ_2'/ρ_1, even though the shock velocity is different in the new frame of reference. The change in the shock velocity is du_1, and therefore

$$\partial V/\partial u_1 = 1. \tag{23}$$

The above equation is exact, while the other two partial derivatives are only first-order approximations.

4. Verification of the shock front equation

The partial derivatives have been calculated, and now the integral expression for the shock velocity as a function of position in a general flow field can be given

$$V(X_1) = V(X_0) + u_1(X_1) - u_1(X_0) + \int_{X=X_0}^{X_1} f(\rho_1, p_1, u_1, V)\, d\rho_1 + \int_{X=X_0}^{X_1} h(\rho_1, p_1, u_1, V)\, dp_1. \tag{24}$$

Equation (24) is the most general possible equation, which can be used to describe the motion of a shock wave through an inhomogeneous medium, since no assumptions are made about specific relationships amongst the downstream variables. Of course, in many experiments such relationships will exist and can be used. As a simple application of the shock front equation, we have used it to predict the local variation in shock front velocity through a known rarefaction wave. In order to generate both the rarefaction wave and the shock wave, we have spark-ignited a plane linear detonation in equimolar acetylene-oxygen mixtures at an initial pressure of 200 Torr. The Chapman–Jouquet detonation proceeds at a constant velocity of 2·8 km/sec, and is immediately followed by a rarefaction wave. The detonation wave is incident upon a reflector, and a shock wave is reflected back through the rarefaction wave. This shock should obey the shock front equation (24). We assume that a secondary reaction does not occur in the reflected shock, because all of the oxygen should be used up in the initial detonation.

The distribution of the particle velocity, density, and pressure through the rarefaction wave have been given by Taylor (1950). The particle velocity is linear along the characteristics and is zero at the rarefaction tail. The density and pressure distributions are found assuming

$$p\rho^{-\gamma} = \text{constant},$$

and are given by
$$\frac{\rho}{\rho_0} = \left(1 - \frac{\gamma-1}{2}\frac{|u|}{a_0}\right)^{2/(\gamma-1)}, \tag{25}$$
$$\frac{p}{p_0} = \left(1 - \frac{\gamma-1}{2}\frac{|u|}{a_0}\right)^{2\gamma/(\gamma-1)}, \tag{26}$$
γ is taken to be constant, $\gamma = 1{\cdot}2$ (Pearson & Fellinger 1965).

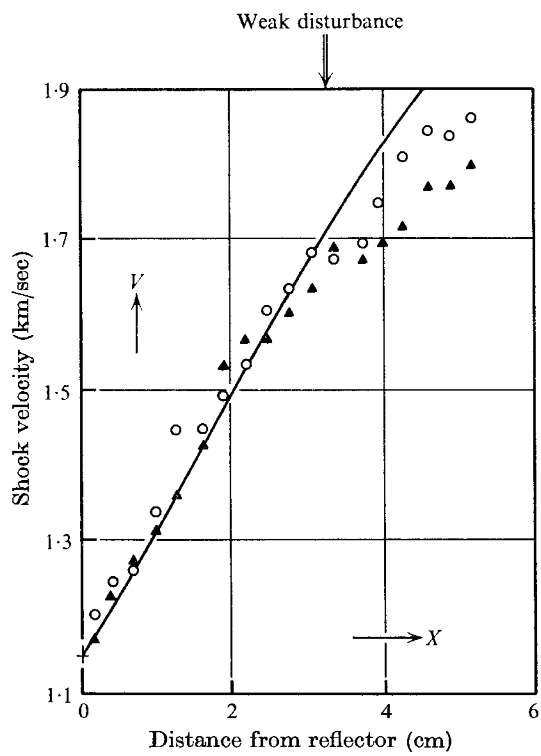

FIGURE 3. The shock velocity is plotted against the fractional distance through the rarefaction wave. The points are measured values from two separate smear pictures under identical conditions. The line represents the predicted shock velocities using the shock front equation. The arrow indicates where the weak compression wave intersects the reflected shock.

The shock velocity was calculated iteratively as a function of position, using a simple predictor numerical integration formula for (24), and using the measured value of 1·15 km/sec for the starting velocity immediately at the reflector. The experimental and theoretical shock velocity profiles (figure 3) agree remarkably well, until a weak disturbance intersects the reflected shock. This disturbance is generated at the ignition end of the tube, and can be seen clearly in the smear picture (figure 4, plate 1). We expect that this disturbance is a weak compression wave, which changes the pressure, density and particle velocity distributions ((25) and (26)) through the rarefaction wave. Thus, the measured shock velocity values verify the shock front equation (24) as long as the initial parameters are known.

To further test the shock front equation, the gas mixture was changed by mixing various amounts of inert gases (argon and helium) into the equimolar acetylene-oxygen. The detonations now proceeded at a different velocity, which implies that a different, but still known, rarefaction wave was created. The initial pressure was increased to 300 Torr to ensure that a detonation would still be produced. The values for the detonation velocity, initial reflected shock

Gas	γ	Detonation velocity (km/sec)	Initial shock velocity (km/sec)
25 % He	1·32	2·84	0·95
15 % Ar	1·27	2·36	0·85
33 % Ar	1·36	2·26	0·78
50 % Ar	1·44	2·15	0·76

TABLE 1

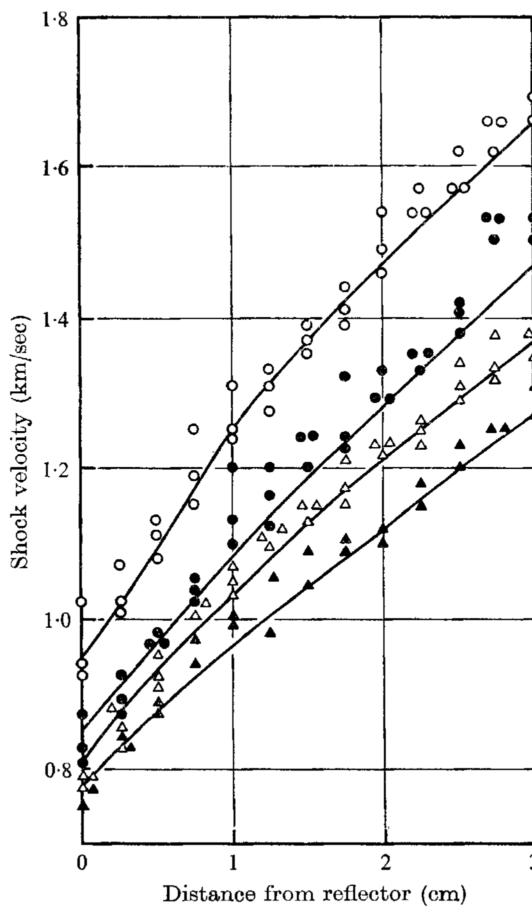

FIGURE 5. The shock velocity is plotted against the distance through the rarefaction wave. The points are values measured from smear pictures. The lines represent the predicted shock velocity using the shock front equation. ○, 75 % $C_2H_2-O_2$ and 25 % He at 300 Torr. ●, 85 % $C_2H_2-O_2$ and 15 % Ar at 300 Torr. △, 67 % $C_2H_2-O_2$ and 33 % Ar at 300 Torr. ▲, 50 % $C_2H_2-O_2$ and 50 % Ar at 300 Torr.

velocity and the adiabatic constant appear in table 1. The experimental values for the shock velocity (figure 5) again agree with the values predicted by the shock front equation.

5. Summary

We have derived and experimentally verified an integral expression for the shock velocity as a function of position in a general flow field:

$$\begin{aligned} V(X_1) = {} & V(X_0) + u_1(X_1) - u_1(X_0) \\ & + \int_{X=X_0}^{X_1} \frac{V-u_1}{2}\left(1 - \frac{Z-1}{2(\lambda^2+Z)\left(1-\dfrac{Z-1}{2(\lambda^2+Z)} + \left[\dfrac{1}{Z}\dfrac{1+\lambda^2 Z}{1+\lambda^2}\right]\right)^{\frac{1}{2}}}\right)\left(\frac{dp_1}{p_1} - \frac{d\rho_1}{\rho_1}\right) \\ & - \int_{X=X_0}^{X_1} \frac{(V-u_1)\left[Z\dfrac{1+\lambda^2 Z}{1+\lambda^2}\right]^{\frac{1}{2}} dp_1}{2p_1(\lambda^2+Z)\left(1-\dfrac{Z-1}{2(\lambda^2+Z)} + \left[\dfrac{1}{Z}\dfrac{1+\lambda^2 Z}{1+\lambda^2}\right]^{\frac{1}{2}}\right)}. \end{aligned}$$

This shock front equation can be used to predict the local variation in shock front velocity through a completely general inhomogeneous flow field where the initial pressure, density, and particle velocity are known. It may also be useful in understanding certain astrophysical phenomena, such as colliding stars and the emission of mass from the surface of stars due to shock waves. The shock front equation also governs the motion of shock waves through the earth's atmosphere.

This work was supported by a research grant of the Atomic Energy Control Board of Canada. The authors are grateful to Professor R. F. Chisnell for his interesting comments on the manuscript.

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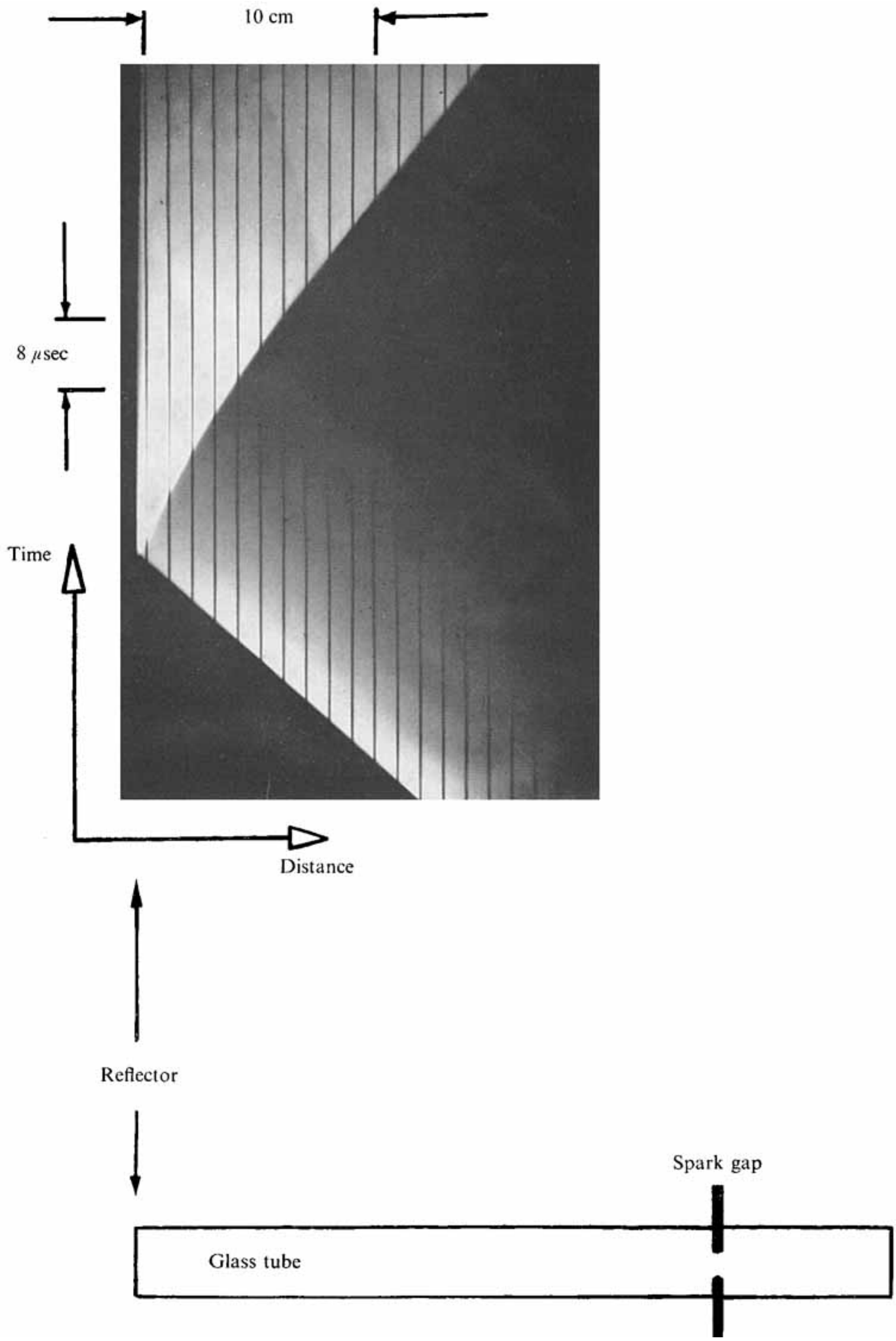

FIGURE 4. This smear picture shows the complete detonation and the reflected shock being accelerated through the rarefaction wave. The gas is $C_2H_2 + O_2$ at 200 Torr. The weak compression wave which destroys the rarefaction wave equations can be clearly seen behind the detonation and looks almost like a characteristic of the rarefaction wave. The apparatus is sketched.